Stochastic Asset Generators for Investment Portfolios of Bonds, Stocks, Real Estate, Commodities, and Foreign Assets Based on the Double Mean Reverting Process™ and the Vector Autoregressive Diffusion.

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ABSTRACT

This paper considers a stochastic process on asset returns and stochastic asset return generators on stock indices, bond indices and other stock like indices such as real estate. It gives a model of the joint interest rate and investment indices process. This n-dimensional model can be used to model stock like indices such as the TSE 300, a managed stock index, an index invested in stocks and bonds, a foreign investment index, or all of these together. The n-dimensional process allows for having correlations between the different indices, as well as modeling inter-relationships in the conditional expected return based on the elements of the state vector modeled in the process. Commodities and other investment categories like junk bonds can be modeled as equity like indices.

This generator can be applied to modeling guarantees on investment products in any of the asset categories. It can be used to model these guarantees when the customer has the ability to change the investment mix over time or to switch the investment mix. This generator can also be used to analyze the asset allocation decision for a company's investment portfolio.

The most difficult part in developing a stochastic asset generator for this set of investments is developing the interest rate model. The interest rate model must be arbitrage free, i.e. no free lunches, and this arbitrage free process must be estimated in the real and risk neutral probability measures. The model must be sufficiently stable and robust in both the real and risk neutral probability measures that it can be applied over long time intervals.

We first review the interest rate modeling portion of the paper. This covers the general issues in interest rate models that we think are appropriate as well as the application to the Canadian economy.

The Double Mean Reverting Process™ (DMRP™) Economic Scenario Generator™ (ESG™) was calibrated to the Canadian economy. . In addition to being arbitrage free, the DMRP™ ESG™ has the capability to generate all interest rate values along the entire yield-to-maturity curve at any time interval required. Its building blocks include interest rates using both risk-neutral and real probabilities, historical calibration of parameter groups, and adjustment to current market conditions. The DMRP™ uses two variables: 1) the logarithm of the overnight rate and 2) the target rate towards which the logarithm of the overnight rate mean reverts to. With these two variables, the ESG™ model can accurately quantify the length and duration of low rates, the length and duration of extreme shapes very steep or flat to inverted yield curves, and the volatility

of rates, to include the tendency to move towards those situations. The ESG™ achieves these listed objectives by controlling the following model characteristics:

1) Speed of mean reversion of level of rates from low or high levels to normal levels
2) Speed of mean reversion of yield curve shape from extreme levels to normal levels
3) Volatility of short and long term rates
4) Volatility of long rates minus short rates.

DEFINITIONS

<u>Arbitrage-Free</u> Arbitrage-free means that there are no arbitrage opportunities in some set of prices at one or more dates. Arbitrage opportunity is defined as the ability to make zero net investment, have no probability of loss, and have a positive probability of gain.

<u>Risk Neutral Probability</u> For an arbitrage-free model with certain additional technical assumptions, and without regard to any calibration to market prices, there exists a risk neutral probability for the random events in the model, which satisfies the following condition. For any liability whose random cashflows are spanned by the random variables of the arbitrage-free model , the amount of assets needed to exactly pay off the liability cashflows at all points in time under any circumstances for any given dynamic asset investment strategy equals the expected value of the discounted liability cahsflows under the risk neutral probability. There exists a risk neutral probability only if the model is arbitrage-free. Note that this has nothing to do with calibration.

<u>Real Probability</u> Real probability is the actual set of probabilities (according to a model) of future events. Real probabilities are defined for the state variables of the model and also all market prices, or events defined in terms of market prices. An example of an event is the inversion of 3-month and 10-year yields at a specified future date. Real probability is difficult to estimate, because it requires a robust model of historical asset returns.

Chart 14.

Table 1.

CANADIAN VS. U.S. HISTORICAL INTEREST RATES

UNITED STATES

Means

	3-month	6-month	1-year	2-year	3-year	5-year	10-year	30-year
Jun82 - Dec91	0.0755	0.0785	0.0814	0.0867	0.0888	0.0916	0.0947	0.0956
Jan92 - Sep98	0.0467	0.0486	0.0509	0.0557	0.0582	0.0621	0.0660	0.0699

Standard Deviations

	3-month	6-month	1-year	2-year	3-year	5-year	10-year	30-year
Jun82 - Dec91	0.0153	0.0164	0.0172	0.0179	0.0180	0.0181	0.0175	0.0167
Jan92 - Sep98	0.0098	0.0100	0.0102	0.0094	0.0089	0.0082	0.0080	0.0077

CANADA

Means

	3-month	**6-month**	**1-year**	**2-year**	**3-year**	**5-year**	**10-year**	**30-year**
Jun82 - Dec91	0.1014	0.1030	0.1042	0.1020	0.1017	0.1030	0.1055	0.1079
Jan92 - Feb99	0.0513	0.0537	0.0567	0.0594	0.0626	0.0662	0.0714	0.0759

Standard Deviations

	3-month	**6-month**	**1-year**	**2-year**	**3-year**	**5-year**	**10-year**	**30-year**
Jun82 - Dec91	0.0171	0.0168	0.0160	0.0134	0.0132	0.0133	0.0141	0.0148
Jan92 - Feb99	0.0144	0.0139	0.0135	0.0129	0.0126	0.0123	0.0123	0.0121

NOTE: The Canadian data covers a time period that goes through 2/99, while the United States data only goes through 9/98.

APPENDIX B. INTEREST RATE GENERATORS- EVAULATION

When applying an interest rate generator model to Asset Liability Management (ALM), a model is only as good as its statistical validity. Models that require re-calibration with each new day's data are not a valid basis of pricing or risk analysis. If the model itself is arbitrage ridden, then an ALM simulation model will trade on the discrepancies. Arbitrage free models that are robust and calibrated to history are a good but still imperfect basis of ALM.

Measures that come out of arbitrage free models and ALM models are Market Value of Liabilities, Economic Reserve of Liabilities, Expected Return on Equity, Distribution of Distributable Earnings, Fair Value of Derivative Prices, and Duration with respect to each stochastic factor.

Decisions that come out of these models are Asset Allocation, Derivative Purchases, Level of Risk Capital, Investment Level in a Business, and Pricing and Structuring of Liabilities.

When evaluating Interest Rate Generators, you must apply the criteria previously discussed:

1. Is it arbitrage-free?

2. Does it generate both risk-neutral and real interest rate scenarios?

3. Does it have a common set of yield curves and yield curve sequences for both risk-neutral and real, but a different set of probabilities?

4. Does it maintain the distinction between fixed parameters calibrated to historical periods and random variables refit daily?

5. Are realistic probabilities calibrated for a long comprehensive time period, say 25 years or longer?

6. Can it reproduce yield curves over a long period, say 20 years, without adjusting its fixed parameters?

7. Can it reproduce yield curves for both real and risk-neutral for periods of 20 years?

8. In comparing yield curve generators, how different are the yield curves and yield curve sequences, as opposed to the probabilities of the sequences?

9. How does the model’s frequency distribution in the range of 2 to 20 percent compare to the actual, observed data in the last 20 to 40 years.

10. How does the model’s frequency of yield curve inversions compare to the historical observed frequency.

Given that the model conforms to the above list of questions, now let us conduct a general evaluation of an interest rate generator system by examining the following questions:

1. Is its historical calibration updated and maintained either externally or internally to the model?

2. If internal, are tools provided for this work in the model or can be built internally at reasonable cost?

3. Are critical issues addressed, such as probability of high and low interest rate scenarios needed to determine probability of ruin, risk/reward tradeoffs, and costs of guarantees and options?

4. Is there sufficient depth of model sophistication, historical research, and discussion material to support institutional cost of education for use and application of the system?

5. Is it robust such that business decisions made on it are likely to remain stable as interest rate generator systems and their historical calibrations evolve over the next 5 years?

Evaluation of One Factor Models

Using the above criteria, the general weaknesses of One-Factor Models can be seen. One-factor models are unable to represent the interest rate process. They can not reproduce the qualitative stylized facts. They can not reproduce the quantitative stylized facts. They can not match the yield curves from 1970 to 1994 by just varying one random factor. They can not correctly identify expected return as the yield curve varies. They can not correctly represent the risks of yield curve shapes, inversions, and sequences. They can not correctly represent the risk of rates persisting at high or low levels for a sustained period.

Consequently one factor models do not allow for the elements of good portfolio management, which are: 1) Assessment of portfolio expected return, 2) Assessment of portfolio risk, and 3) Proper portfolio strategy development.

For pricing applications, One-Factor Models are limited in these areas. One-factor models do not correctly assess the value of caps and mortgages whose value depends on the length of time that rates persist at high or low levels. Corporate bonds have imbedded options whose value depends on the probability rates can reach the optimal exercise boundary and for which the optimal exercise boundary in turn depends on whether rates are likely to persist at the level of the boundary, immediately move back towards the normal level, or have a likely chance to move farther. One-factor models can not evaluate this complex problem accurately. Other more complex securities whose cashflows or options are dependent on the characteristics or the interest rate process as to the likelihood of yield curves and configurations, and their likely persistence.

Evaluation of Some Two Factor Models

1. Brennan and Schwartz: This model has recently been shown by Hogan [10] to have mathematical inconsistencies.

2. Two factor Cox, Ingersoll, and Ross: This model requires that the two factors be uncorrelated, which is not valid and limits the ability to calibrate the model historically.

3. Making the second factor a stochastic volatility variable, which is discussed below.

4. Two factor Heath, Jarrow, and Morton (HJM) models: These models as provided currently are not historically calibrated and do not provide realistic scenarios, but only risk neutral scenarios.

5. Double Mean Reverting Process™, which fulfills the criteria outlined above.

There are inherent weaknesses in two factor models, which use stochastic volatility as the second factor. Stochastic volatility has a secondary effect on the prices of bonds under an interest rate model. Consequently, the yields are little affected as well. Therefore, as a model of the yield curve, these models are really one-factor models. They are subject to all the criticisms of one-factor models.

There are inherent weakness in the Heath, Jarrow, and Morton (HJM) Models. These models contain numerous parameters fit to an initial yield curve, and assumed constant by the model. However, these parameters are in fact not stable. Furthermore, HJM models are all risk neutral. They can not be used to generate scenarios for cashflow models without substantial further work by the user to develop the realistic version. HJM models were billed as having the advantage of not separately identifying the market price of risk so that one did not need to identify the market price of risk to price securities. This supposed advantage is a clear disadvantage for portfolio evaluation and scenario generation since without identification of the market price of risk the model can not be used for these purposes. In conclusion, by design and by their very nature, HJM models are not easily applied for scenario projection for cashflow models and for evaluating risk and return of securities.

Finally, let us lay out both the strengths and weaknesses of the DMRP™ Interest Rate Model.

<u>Strengths of the DMRP™ Interest Rate Model</u>

1. Able to reproduce range of historic yield curves.

2. Reproduces tendency of rates to persist in a trading range for an extended period and then switch to a new trading range.

3. Models yield curve inversions and other stylized facts.

4. Can reproduce historical frequency of yield curves, even if that frequency is later modified based on judgement or other factors.

5. Model lends itself to sensitivity analysis to test other sets of assumptions.

<u>Weaknesses of the DMRP™ Model</u>

1. Volatility of short-term rates is not stochastic.

2. Difficult to represent all historical episodes with one calibration. For example, including the extended low rates of the Great Depression is difficult to do at the same time including the last 25 years.

3. Even with the greater stability of the DMRP™ model, there is still slow variation in the fixed parameter values.

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