

Discrepancy and Discrepancies in Monte Carlo

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Chapter 1

For users of Monte Carlo

1.1 Introduction

I – 1 Monte Carlo in Finance

Monte Carlo was introduced into finance for the valuation of securities by Boyle in 1977 [9]. Since then it has been widely used for both valuation and risk analysis.

I – 2 Option Pricing

Option pricing formulas were developed by Bachelier [2], Kruizenga [38], Sprenkle [53] and Boness [7]¹. Black-Scholes [6] showed how to derive a formula of the same form as Boness [7] using an equilibrium approach. Boness and the others had been unable to come up with the right discount rate and expected rate of return, the risk free rate in both cases, these were discovered and proven by Black-Scholes. Black-Scholes report a no-arbitrage derivation based on a suggestion by Merton who showed the Boness type formula with the Black-Scholes parameter restrictions for equity options also obtains if interest rates are normally distributed. Black-Scholes showed how to derive a partial differential equation of the McKean-Samuelson type [46] [52]. Merton [47] extended that equation to include a second random source from interest rates for pricing equity options only. Cox-Ross [18] showed how to interpret the Black-Scholes solution in terms of risk neutral probability and to price options with other stock price processes. This was an economic interpretation of the mathematics of Black-Scholes and Merton which was already risk neutral probability.

¹Notes in the next few sections are based on those in Davlin et al. Tenney [20]

The general approach to arbitrage in a single currency was done by Garman [24] at the same time as Richard developed it for just random interest rates [51].² The Garman approach was for any type of security contingent on any type of random variable.

I – 3 Interest Rate Models

Monte Carlo is the leading method used to price mortgages and equilibrium mortgage interest rates are determined by the use of Monte Carlo simulation models. The mortgage applications use interest rate models as the main random factor. One factor interest rate models under equilibrium were developed by Vasicek [56] and Cox, Ingersoll and Ross [17]³. Richard [51] and Cox, Ingersoll and Ross [17] extended the one factor CIR model to two factors independently.

The key base of modern multi-factor interest rate models with closed form solutions are the multivariate normal models of Langetieg [39] and the two factor square root model of Richard [51] and Cox, Ingersoll and Ross (CIR) [17] .

Special cases of Langetieg’s model are Ho-Lee, , [31], Jamshidian, [34] [35] [36] , and Hull and White and [32] .

Later models built on Langetieg include the quadratic model Longstaff [41], Beaglehole et al., [4] [5] , Jamshidian [37] ⁴ Eterovic [23], Ahn, Dittmar , and Gallant [1], Leippold and Wu [40] and others, Lin Chen’s [13] 3 factor model and the affine model of Duffie and Kan [21], and the Heath, Jarrow, and Morton (HJM) methodology [30].

Langetieg derived a Boness-like formula for options on stocks with his interest rate model applying Merton’s argument on Black-Scholes. Hull and White showed that Langetieg’s approach could be slightly modified to apply to their version of the Langetieg model. State prices or option prices for these models were developed by Cox, Ingersoll and Ross [17] Jamshidian [35], Longstaff [42], Hull and White [32], Beaglehole et al. [4], Milne and Turnbull [55], Chen and Scott [14], Longstaff and Schwartz [43], Constantinides [15], Lin Chen [13], Manistre [45], Duffie, Pan and Singleton [22] and others. Earlier work on state prices or Green’s functions in finance traces back to McKean [46], Garman [24], Cox, Ingersoll and Ross [16], Breeden and Litzenberger [11], Banz and Miller [3], Ingersoll [33], and Merton [48]. In addition, Lucas [44] derived state prices building on the foundation of

²This was in a 1976 Carnegie Mellon working paper.

³Both were working papers in 1976

⁴See Manistre [45] for a general discussion of multifactor models.

Hakansson's [27], [28] approach to optimal savings, consumption and portfolio choice. Lucas developed a state price using marginal rates of substitution that arise in the Hakansson type optimizations of consumption, savings and portfolio decisions in a multiple-period context. Cox, Ingersoll and Ross and Vasicek's equilibriums are also built on the Hakansson foundation.

Because of the prepayment models and the frequent use of interest rate models without closed form solution, Monte Carlo and Low Discrepancy Sequences are often used for mortgage pricing and risk analysis. An example of a model without a closed form is the DMRPTM⁵ model, see for example Groover [25], Groover et al. [26] and Craighead et al. [19].

I – 4 Insurance Applications

Monte Carlo has been applied to insurance products or asset liability management by Boyle, Brender, Brown, Craighead, Embrechts, Fitton, Hancock, Hardy, Manistre, and Panjer in numerous papers and actuarial studies to name but a few. It has been used to set Segregated Fund Guarantee regulations in Canada. It has been used for interpreting CARVM for Equity Indexed Annuities, for C-3 Phase One risk based capital for interest rates, C-3 Phase Two for variable annuity guarantees, and other studied by the American Academy of Actuaries and Canadian Institute of Actuaries .

Monte Carlo has been applied to guarantees on segregated funds by Mary Hardy using a regime switching approach [29]. Error bounds for this model or related ones can be calculated with discrepancy. Low Discrepancy Methods or Quasi-Random Monte Carlo can be combined with that model for analyzing insurance problems such as variable annuity guarantees including reserve and capital requirements.

I – 5 Monte Carlo

Monte Carlo was developed at Los Alamos in the 1940's for calculations related to nuclear physics. It has since been applied to a variety of problems from analyzing scattering experiments in accelerators to financial applications.

I – 6 Quasi-Random Monte Carlo

Quasi-Random Monte Carlo means deterministic sequences or points. They are usually chosen because they have a lower discrepancy than "randomly"

⁵Trademark Mark Tenney and Mathematical Finance Company.

chosen points. Paskov and Papageorgiou [50] review the history of applying Quasi-Random Monte Carlo (QRMC) to high-dimensional problems in finance. The text by Niederreiter [49] is one of the classic texts on QRMC and is the basis of most of the theorems presented later.

Some recent applications of QRMC have been made by Boyle, Broadie and Glasserman [10] and Boyle and Tan [8].

1.2 Discrepancy

We consider some informal definitions or characteristics of discrepancy in this section. Later we give the formal mathematical definition.

Definition 1 (Discrepancy) *Discrepancy is a measure of the departure from the uniform covering of a set of numbers in a box in n -dimensional space.*

There are several different measures of discrepancy. The most important is called the "Star Discrepancy". A set of N points uniformly spaced from 0 to 1 on the line have a Star Discrepancy of $\frac{1}{2N}$. This goes to 0 as N goes to infinity. This set of points is a uniform finite covering of the interval 0 to 1. As N goes to infinity, it becomes, for certain integration purposes, a uniform covering of the interval and its Star Discrepancy goes to zero.

II – 1 Variation of a Function

Definition 2 (Variation) *The variation of a function over an interval measures the total up and down height the function experiences added together. In n -dimensions, this definition is generalized to measure a generalization of this concept to n -dimensional surfaces and their "faces".*

1.3 Discrepancies

III – 1 One simulation method is as good as another?

Fallacy 1 *One simulation method is as good as another.*

Counter Statement 1 *The error bound on a method is based on the Star Discrepancy of the points used. The lower the Star Discrepancy the lower the error bound.*

The error bound is the product of the Star Discrepancy and the Variation of the function over the relevant range of the function. Here the function is the product of some function like loss times the probability density function of the loss, if known, or of the variables that drive the loss, the more usual case.

III – 2 Simulation methods are black boxes?

Fallacy 2 *Simulation methods are black boxes to be used on a trial and error basis.*

Counter Statement 2 *Simulation is part of the theory of measure, integration and probability. There are basic theorems about the accuracy of integrals that apply to any set of points used in simulation to calculate an output that corresponds to an integral. Discrepancy is an inherent part of the theory of such integration.*

III – 3 Error bound of Monte Carlo is σ^2/N ?

Fallacy 3 *The error bound of a Monte Carlo is σ^2/N where N is the number of scenarios and σ is the standard deviation of some output like loss.*

Counter Statement 3 *This is the expected error and is not an error bound. The actual error in a Monte Carlo simulation can be greater than this. If the output variable is unbounded, the error in any set of Monte Carlo Scenarios is unbounded as well.*

The Monte Carlo may technically be bounded because of the finite size of the generator of pseudo-random numbers where those are used. A perfect random number generator results in no bound on the error when the loss or output variable is unbounded. An example is a call option on a stock. Its payoff is unbounded. Monte Carlo estimation of its average payoff or even its price has an error that is not bounded prior to the simulation.

III – 4 Discrepancy has no relation to Monte Carlo?

Fallacy 4 *If you use regular Monte Carlo, discrepancy doesn't have anything to do with it.*

Counter Statement 4 *The error bound on use of Monte Carlo is in terms of the Star Discrepancy of the random set of points generated by monte carlo. It doesn't matter how the points are generated, their Star Discrepancy determines the error bound.*

However, if the function does not have finite variation, then the Koksma-Hlawka Inequality doesn't apply.

III – 5 Low discrepancy just another tool?

Fallacy 5 *Low Discrepancy is just another tool in the toolbox of simulation.*

Counter Statement 5 *Discrepancy establishes the error bounds on any set of points whatever the method of their simulation. Methods provide superior error bounds by having lower Star Discrepancy.*

III – 6 Better error bounds than low discrepancy can be invented?

Fallacy 6 *Better error bounds than low discrepancy can be invented.*

Counter Statement 6 *Theorem 2.12 of Niederreiter shows that no better error bound than the Koksma-Hlawka Inequality exists. This means there can't be a better error bound for all functions of finite variation.*

III – 7 Better methods than low discrepancy can be invented?

Fallacy 7 *A general purpose method better than low discrepancy could be invented.*

Counter Statement 7 *Theorem 2.12 of Niederreiter shows that no better error bound than the Koksma-Hlawka Inequality exists. This means that improving on the maximum error means reducing the discrepancy.*

A specific purpose method might have advantages however.

III – 8 With Simulation Any Method Goes?

Fallacy 8 *You just choose whatever method you want to do simulation, there is no theoretical reason to prefer one over the other.*

Counter Statement 8 *There are theoretical reasons and discrepancy is one of the key measures in choosing one over another. Discrepancy is the best possible error bound. It is really part of the theory of integration with finite point sets.*

Just as much as the error analysis of Riemann and Lebesgue Integrals, Discrepancy is a fundamental part of the theory of integration. So is the Variation of the function being integrated. The product of the Variation of the function and the Star Discrepancy is the error bound on any approximation of an integral by a finite set of points. This applies in any number of dimensions. The Koksma-Hlawka Inequality applies in any dimension and the Koksma Inequality applies in one dimension.

III – 9 We don't compute integrals?

Fallacy 9 *We don't compute integrals.*

Counter Statement 9 *Yes you do. Conditional Tail Expectation, CTE, is an integral. Any tail loss is an integral. Percentile is an integral.*

1.4 Cajun Wisdom

IV – 1 Its the Discrepancy

Cajunism 1 *Its the Discrepancy*

IV – 2 Part of Integration Theory

Cajunism 2 *Discrepancy is part of integration theory.*

IV – 3 Its all integration

Cajunism 3 *Almost every statistic out of a random simulation is the integral of something.*

The percentile equals the integral of an indicator function, that some output variable, like loss, is greater than or less than some value. Take a percentile like the 90-th percentile on a test graded from 0 to 100. Say it is 85. So 10 percent of the class scored higher than 85. Suppose this was the true population distribution, not just the sample distribution. The indicator function for this is the score is greater than 85. The integral of this indicator function is .1, i.e. 10 percent.

We normally think of percentile a little bit differently, to set up this indicator function we already have to know the percentile. However, it is an easy way to see that the percentile corresponds to the integral of some indicator function. Here the integral is over the probability of sample

outcomes, here the test score of an individual. We imagine we have an infinite population indexed by z , say and we integrate z over its range, the index set, which could be all real numbers from 0 to 1, corresponding to an uncountable population.

IV – 4 CTE for segregated fund guarantees is a good application

Guarantees for segregated funds typically have bounded payoff. For these types of guarantees, the variation of the function being integrated is usually finite so that the error in the integral is bounded by the Star Discrepancy times the variation of the function. This follows from the Koksma and Koksma-Hlawka inequalities.

Cajunism 4 *Conditional Tail Expectation for segregated fund guarantees usually involves a bounded payoff, similar to a put option. The variation is therefore often finite. This means the inequality bounds for any Monte Carlo or Quasi Random Monte Carlo method apply. This in turn means reducing the error bound means lowering the discrepancy of the points used to perform the integral.*

IV – 5 Discrepancy measures an integration method

Cajunism 5 *Discrepancy is part of the theory of integration. When it is applicable, it is the way to measure the guaranteed maximum error.*

IV – 6 QRMC-LDS provides value even for small sets of points

Cajunism 6 *Experience in applying QRMC-LDS methods to insurance and financial problems has shown they add value to a wide range of problems. This is true even when the number of scenarios or points used is small.*

IV – 7 Variation on puts

Cajunism 7 *For European puts on stocks the Variation is the strike price.*

Cajunism 8 *For European puts on other instruments the Variation is at least the strike price.*

IV – 8 Set up with Bounded Variation

Cajunism 9 *If possible, compute integrals on puts or guarantees. Use Put-Call Parity where possible to get bounds on call option errors.*

Cajunism 10 *If the total payments are bounded, the variation is often bounded as well. Try to set up your model or application so you calculate something whose value is bounded. Note that the probability density function is included in what is integrated which often helps.*

Cajunism 11 *Many insurance applications involve bounded payoffs. These often have bounded variation.*

IV – 9 Its the Theorems

Cajunism 12 *Discrepancy is a basic part of the theory of error bounds in numerical integration. The maximum error in the use of a set of points in an algorithm to compute an integral is the Star Discrepancy times the Variation, when the variation is finite. These error bounds can't be improved on for the general case. These are all part of the core theorems of numerical integration and integration theory.*

IV – 10 Experience is good in many cases

Cajunism 13 *Even when the payoff is unbounded so that the classic inequalities don't apply, low discrepancy sequences have proven to be superior in practice in many applications. This includes calculating the price of calls, an unbounded payoff case.*

Chapter 2

Theorems

We follow Chapter 13 of Carothers [12], especially pages 202 to 203 and Chapter 2 of Niederreiter [49]. Their notations and terminology are largely consistent. We follow both closely.

2.1 Discrepancy

In this section, we follow Chapter 2 of Niederreiter [49], especially pages 14 and 19.

The most important Discrepancy is Star Discrepancy.

Definition 3 (Subintervals anchored at zero) *Consider a unit hypercube in s dimensions. Suppose that in each dimension i , the i -th dimension's coordinate is greater than or equal to 0 and less than or equal to u_i . A set of points satisfying these inequalities is a type of sub-interval of the unit hypercube.*

Definition 4 (Star Discrepancy) *Consider a unit hypercube in s dimensions. Consider the sub-intervals anchored at zero, and let B be such a sub-interval. For N points, let P be the set x_1, \dots, x_N . Let $A(B, P)$ be the number of points of P in the sub-interval B . The Star Discrepancy is*

$$D_N^*(P) = \sup \left| \frac{A(B, P)}{N} - \lambda_s(B) \right| \quad (2.1)$$

where $\lambda_s(B)$ is the volume of the sub-interval B .

When we use the term discrepancy, we shall mean Star Discrepancy unless the sense indicates otherwise.

2.2 Variation

II – 1 One Dimension

We follow Chapter 13 of Carothers [12], especially pages 202 to 203 and Chapter 2 of Niederreiter [49]. In this section we use Carothers terminology and notation.

We start with a function f of a single variable over a finite closed interval from a to b . Use the variable t as the independent variable, so t varies from a to b inclusive, $[a, b]$.

Definition 5 (Partition P of $[a, b]$) A set, P , of points is a partition of the set is specified as follows. P is a set of points $t_i, i=0, \dots, n$ such that $t_i < t_{i+1}$, $t_0 = a$ and $t_n = b$.

Definition 6 (Refinement Q of a Partition) A partition Q is a refinement of P , if every point of P is a point of Q .

Definition 7 (Variation for a Partition)

$$V(f, P) = \sum_{i=1}^n |f(t_i) - f(t_{i-1})| \quad (2.2)$$

Definition 8 (Total Variation) The total variation of f over $[a, b]$ is defined by

$$V_a^b f = \sup_P V(f, P) \quad (2.3)$$

Here \sup is the supremum over all possible partitions of the interval $[a, b]$.

II – 2 Theorems on Variation

Carothers Chapter 13 [12] gives theorems on the variation in one dimension. See especially p203-204.

Theorem 1 (Expected Error) If $h: [a, b] \rightarrow \mathbb{R}$ is monotonic, then

$$V(h, P) = |h(b) - h(a)| \quad (2.4)$$

for any partition P of $[a, b]$. It follows that the total variation equals $|h(b) - h(a)|$.

II – 3 Hardy Krause variation n-dimensions

We follow Chapter 2 of Niederreiter [49] page 19. We consider the s dimensional unit hypercube. Let J be a subinterval of the s dimensional unit hypercube. Let $\Delta(h; J)$ be the sum of values of h at the vertices of J with alternating signs at the vertices.

Definition 9 (Vitali Variation) *The variation of h on the unit hyper cube in the sense of Vitali is defined by*

$$V^s(h) = \sup_P \sum_{J \in P} |\Delta(h; J)| \quad (2.5)$$

where the supremum is over all partitions P of the unit hypercube into subintervals.

Definition 10 (Hardy-Krause Variation) *The variation of h on the unit hyper cube in dimension s is*

$$V(h) = \sum V^s(h) \quad (2.6)$$

where the sum is over the "faces" of the unit hypercube in s or lower dimensions.

2.3 Monte Carlo Expected Error

We follow Chapter 2 of Niederreiter [49] page 4. The notation is changed somewhat so as to reduce name clashes with later applications in finance.

Consider a function $h(x)$ where x is an n -dimensional vector over some volume Ω .

$$\int h(x) dx = \lambda(\Omega) \int h d\phi = \lambda(\Omega) E[h] \quad (2.7)$$

where $d\phi = dx/\lambda(\Omega)$ and $\lambda(\Omega)$ is the volume of Ω , i.e. its Lebesgue measure. This is equation 1.3 of Niederreiter [49] page 4. Note that h need not be non-negative or positive.

We have the expression

Definition 11 (Monte Carlo Estimate)

$$E[h] = \frac{1}{N} \sum_{i=1}^N f(x_i) + \epsilon \quad (2.8)$$

This becomes an approximation when we assume that ϵ is zero. What makes it Monte Carlo is when the points x_i are distributed according to the distribution ϕ , where now we don't require that we have the ϕ above for Niederreiter's explanation of how to write an integral in terms of an expectation.

The variance of the distribution is defined as

Definition 12 (Variance)

$$\sigma^2(h) = E[(h - E(h))^2] \quad (2.9)$$

The expected error of the Monte Carlo estimate is defined to be its sample variance, i.e.

Definition 13 (Expected Error)

$$\sigma_e^2(h) = E\left[\left(\frac{1}{N} \sum_{i=1}^N f(x_i) - E(h)\right)^2\right] \quad (2.10)$$

Here we use N as the divisor and not $N-1$ following Niederreiter.

Niederreiter proves Theorem 1.1 (his numbering) on (his) page 4 that

Theorem 2 (Expected Error)

$$\sigma_e^2(h) = \sigma^2(h)/N \quad (2.11)$$

This shows that this expression for the error is the expected error and not derived, at least in this form as a bound on the error.

In fact, it is easy to show examples in which the error is greater than this value. One can simply take a sample in which all the points exceed some lower bound or are lower than some greater bound and choose the bound so that the Monte Carlo Estimate is more than the Expected Error from its expected value.

2.4 Koksma's Inequality

Page 18 of Niederreiter, his Theorem 2.9 states Koksma's inequality

Theorem 3 (Koksma's Inequality) *If h has bounded (total) variation on the closed interval $[0,1]$, then for any points x_1, \dots, x_N , in the interval $[0,1]$, we have*

$$\left| \frac{1}{N} \sum_{i=1}^N h(x_i) - \int_0^1 h(u) du \right| \leq V(h) D_N^*(x_1, \dots, x_N) \quad (2.12)$$

Koksma's inequality applies to where the integral is over one dimension. This applies for example to the Boness [7] formula for the European Put, or special cases of it like the Black Scholes formula.

2.5 Koksma-Hlawka's Inequality

Page 19 of Niederreiter, his Theorem 2.11 states Koksma-Hlawka's inequality.

Theorem 4 (Koksma-Hlawka's Inequality) *If h has bounded (total) variation on the unit hypercube in s dimensions, then for any points x_1, \dots, x_N , in said unit hypercube, we have*

$$\left| \frac{1}{N} \sum_{i=1}^N h(x_i) - \int h(u) du \right| \leq V(h) D_N^*(x_1, \dots, x_N) \quad (2.13)$$

where $V(h)$ is the Hardy-Krause Variation in s dimensions of the function h on the unit hypercube in s dimensions. Here the integral is over the unit hypercube in s dimensions.

The Koksma-Hlawka inequality extends the Koksma inequality to s dimensions. This is what applies to many practical problems in finance and insurance. However, we may not have that $V(f)$ is bounded, especially if the payoff is unbounded.

2.6 Niederreiter Theorem 2.12

Niederreiter Theorem 2.12 (his numbering) proves that the Koksma-Hlawka inequality is the best possible inequality when the function h may come from a broad set of functions. The function space he considers is the set of functions with Hardy-Krause variation equal to 1 and which have continuous derivatives of all orders. The functions can depend on s dimensions and are defined on the unit hypercube in s dimensions. The value of the function is a real number.

Theorem 5 (No go theorem to beat Koksma-Hlawka) *For any points x_1, \dots, x_N in the unit hypercube in s dimensions, and any $\epsilon > 0$ there exists a function h that has Hardy-Krause variation equal to 1 on the unit hyper*

cube in s dimensions, and which has continuous derivatives of all orders such that

$$\left| \frac{1}{N} \sum_{i=1}^N h(x_i) - \int h(u) du \right| > V(h) D_N^*(x_1, \dots, x_N) - \epsilon \quad (2.14)$$

where $V(h)$ is the Hardy-Krause Variation in s dimensions of the function h on the unit hypercube in s dimensions. Here the integral is over the unit hypercube in s dimensions.

We left in the $V(h)$ even though it is 1 to emphasize the result.

What this theorem says is that there is no tighter bound than Koksma-Hlawka when h can be any function in the allowed function space (set of functions). This theorem of Niederreiter and the Koksma-Hlawka theorem show that Discrepancy is a basic part of the theory of integration that within certain specifications can't be improved upon.

2.7 European Put

We show how these methods can be applied to the numerical integration of European Puts. Although this problem was solved by Boness in terms of the cumulative distribution function, reanalyzing this problem from the point of view of discrepancy is a good example of the application of these methods. It is also related to insurance applications in guarantees.

VII – 1 Variation

We apply the above theorems to the European Put. We assume the same set up as Boness and later Black Scholes, i.e. a lognormal process on the stock price S . In this case, we can calculate the variation of the payoff as the strike price K . The Boness Formula, of which the Black Scholes formula is a special case, can be reinterpreted as an integral over u where u varies from 0 to 1. Once this reinterpretation is done, we can prove that the total variation is finite and equal to the strike price.

Let the stock price process be

$$S = e^{\mu t + \sigma \sqrt{t} z} \quad (2.15)$$

where z is a standard mean zero, variance 1 normal. We can write

$$z = F^{-1}(u) \quad (2.16)$$

where F is the cumulative standard normal distribution function. So u varies from 0 to 1. If we write

$$u = F(z) \quad (2.17)$$

it is easier to interpret the relation of z and u . When z is $-\infty$, $F(z) = 0$, so $u=0$. When z is plus infinity, F is 1, so u is 1.

The payoff function for a European put is

$$P = \max[0, K - S] \quad (2.18)$$

This is a monotonic decreasing function of S , i.e. for larger S the value of the put gets smaller or stays the same. The exponential function, standard normal cumulative and its inverse are all monotonic increasing functions, so their compositions are as well. So P is a monotonic decreasing function of u . We can also see check this with a few points. If u is zero, z is minus infinity, so S is zero. In that case, P is K . If u is 1, z is plus infinity, S is infinity, so P is 0. It is clear that as u increases from 0, z increases from $-\infty$. Therefore S increases, therefore $K-S$ decreases, and so P decreases from K to 0. After reaching 0, P stays at 0 as u increases further.

By the theorems in Carothers in Chapter 13, the Put Payoff Function therefore has a variation equal to K , since its maximum value is K and its minimum value 0. When the range of integration covers $S=0$ to $S=K$, the variation is K .

Theorem 6 (Variation of Put) *The Variation of a European put's payoff for the lognormal model is its strike price.*

Discounting this strike price at constant interest rates can only decrease it, so K is still a bound for the variation of such discounted values and therefore for the European put option price.

VII – 2 Koksma Inequality

For the calculation of the European Put in the Boness formula including the special case of Black-Scholes, we have a one dimensional integration. Niederreiter [49] Chapter 3 p23-24 discusses numerical integration in this case. We are integrating u from 0 to 1 to calculate the expected payoff of the put. After that we can discount its payoff at a constant rate in the Boness formula or its Black-Scholes subcase.

If we have N points, the minimum Star Discrepancy is $\frac{1}{2N}$. See Theorem 2.6 of Niederreiter. The minimum is obtained for the choice of points

$$x_i = \frac{2i - 1}{N} \quad (2.19)$$

for $i=1,\dots,N$.

Theorem 7 (Koksma Inequality European Put) *For a European Put in the Boness framework, the numerical integration over the interval of u from 0 to 1 of the Put Payoff Function is given by*

$$|\epsilon| \leq x_i = \frac{K}{2N} \quad (2.20)$$

where K is the strike price. If B is the price of a zero discount bond, the inequality on the price is

$$|\epsilon| \leq x_i = \frac{KB}{2N} \quad (2.21)$$

These inequalities are for the case of direct integration of the Put Payoff Function or Put Price. It is somewhat more efficient to first transform as Boness (and Sprenkle [53]) realized to the form we are familiar with for these integrals in terms of the cumulative normal distribution function.

Chapter 3

Some URLs

In the pdf version the following URL's are active and can be clicked to open up the corresponding web page.

3.1 Boyle

Phelim Boyle originated Monte Carlo in finance for pricing options. He also does research on applying QRMC and LDS in finance and insurance. See his paper for example [10].

<http://www.arts.uwaterloo.ca/ACCT/people/boyle.htm>

3.2 Brender

Allan Brender does research and is an early proponent among regulator in using simulations to set capital for financial institutions. Some of his papers can be obtained at CIA URL's such as

http://www.actuaries.ca/meetings/annual/2002/Handouts/PD-13_Brender.pdf

or http://www.actuaries.ca/meetings/segfund/2001/A_Brender.pdf

3.3 Broadie

Mark Broadie is a researcher at Columbia University in Monte Carlo and QRMC in finance. See for example his paper with Phelim Boyle [10].

<http://www.columbia.edu/~mnb2/broadie/>

3.4 Brown

Robert A. Brown does research in using simulation models for regulation of insurance companies. He is the primary author of the AAA interest rate model for setting capital. A paper of his on the C3 phase 2 project is available at

http://www.actuary.org/pdf/life/c3_march02.pdf

Other papers are available in the SOA library.

<http://www.actuariallibrary.org/>

3.5 Carothers

Neal Carothers has written a very readable book on real analysis [12]. That text was used for the material on Variation of a function of one variable. The theorems he provides were used to prove that the variation of a put option is its strike price for the Boness and Black Scholes assumptions on stock returns.

<http://personal.bgsu.edu/~carother/Vita.html>

3.6 Craighead

Steve Craighead does research in stochastic processes especially when applied to insurance ALM problems. See the paper [19] for estimation of the Double Mean Reverting Process as part of a Vector Auto Regression.

http://www.geocities.com/craighead_steven/index.html

and

<http://www.actuariallibrary.org/>

3.7 Embrechts

Paul Embrechts is does research in stochastic finance and insurance.

<http://www.math.ethz.ch/~embrechts/>

3.8 Hakansson

Nils Hakansson developed the application of dynamic programming to optimal consumption, savings and uncertain lives with stochastic investment opportunity sets. His work on portfolio theory and optimal consumption

and savings is the basis of modern multiperiod equilibrium theories and equilibrium dynamic programming macroeconomics and was one of the core steps in developing the microfoundations of macroeconomics. He along with Markowitz and Sharpe were the trio of Ph.D. theses that form the basis of modern portfolio theory, see [27]. Hakansson's thesis includes the multiperiod extension of Markowitz.

<http://www.haas.berkeley.edu/finance/hakansso.html>

3.9 Hancock

Geoffrey Hancock along with his colleague John Manistre is one of the principal figures in actuarial analysis of segregated fund guarantees in Canada and the United States. Some of his papers can be found at the following URL's. .

<http://www.actuaries.ca/meetings/general/1999/Hancock.pdf>

http://www.actuaries.org/members/en/events/congresses/Cancun/ica2002_author/g/geoffrey_hancock.htm

3.10 Hardy

Mary Hardy has done research in regime switching in insurance. developed and estimated a regime switching stock return model for use in actuarial and insurance applications. Her reports which are one of the most important bases, along with Geoffrey Hancock's work, of recent actuarial regulatory studies in Canada and the U.S. are available through her web page.

http://www.stats.uwaterloo.ca/Stats_Dept/homepages/MRH/MRH.html

3.11 Manistre

John Manistre does research in mathematical finance and insurance. A recent paper of his on partial differential equations applied to Universal Life is available at

<http://www.actuaries.ca/meetings/afir/handouts/Manistre.pdf>

See also

<http://www.actuariallibrary.org/>

3.12 Niederreiter

Harald Niederreiter is one of the early figures in the development of theorems on discrepancy and QRMC and the theory of integration . His text [49] is the basis for much of the material on QRMC and Discrepancy in this paper.

<http://www.dismat.oeaw.ac.at/Niederreiter.shtml>

3.13 Panjer

Harry Panjer is one of the principal figures in bringing together stochastic finance and insurance and applying advanced mathematical methods in insurance. .

http://www.stats.uwaterloo.ca/Stats_Dept/homepages/HHP/HHP.html

3.14 Papageorgiou

Anargyros Papageorgiou does research on high dimensional QRMC and LDS. He is Columbia University's principal researcher on improving FinDer and Low Discrepancy Sequences for high dimensional problems. FinDer is the system maintained by Columbia University for QRMC and LDS.

<http://www1.cs.columbia.edu/~ap/index.html>

See especially the papers:

http://www1.cs.columbia.edu/~ap/html/body_publications.html

3.15 Elias Shiu

Elias Shiu along with Gerber has developed an innovative approach to financial and insurance pricing. .

<http://www.stat.uiowa.edu/~eshiu/>

3.16 Ken Seng Tan

Ken Seng Tan does research in the application of QRMC to finance and insurance. See for example his paper with Boyle [8].

http://www.stats.uwaterloo.ca/Stats_Dept/homepages/KST/KST.html

3.17 Tenney

Leon Tenney does research on the combination of QRMC, vector autoregressions and multifactor interest rate models at Mathematical Finance Company. A previous paper of his with Justin Bobo et al. [54] on these subjects is available at

<http://www.actuaries.ca/meetings/segfund/papers/Tenney.pdf>

and also at

<http://www.mathematical-finance.com/contacts.html>

3.18 Traub

Jospeh Traub was one of the early researchers at Columbia on applying QRMC and LDS to financial problems.

<http://www1.cs.columbia.edu/~traub/>

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